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TORSION OF I-TYPE AND H-TYPE BEAMS

By John E. Goldberg, M. ASCE

STRUCTURAL DIVISION

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AMERICAN SOCIETY OF CIVIL ENGINEERS

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PAPERS

TORSION OF I-TYPE AND H-TYPE BEAMS

By John E. Goldberg,[1] M. ASCE

Synopsis

Beginning with a review of pure torsion and torsion bending of I-type and H-type sections, the basic differential equation is obtained for the twisting of such sections. The stresses resulting from twist—namely, the simple torsional stresses and longitudinal and shearing stresses due to warping restraints—are discussed from the engineering viewpoint. Particular solutions of the differential equation are obtained for various warping conditions at the ends, and it is shown how these solutions are combined to formulate and analyze various problems, including that of a framed floor panel. The concept of the warping angle is introduced in the belief that its easily visualized character leads to a simpler formulation of problems lying within the scope of this theory.

Introduction

In certain important cases of twisting of I-beams and similar types of beams, the bending stiffness of the flanges in their respective planes and any possible additional restraints against warping at any of the cross sections have a significant influence upon the manner in which the torsional load is resisted, and upon the over-all torsional stiffness of the beam.

Consider an I-beam (Fig. 1(*a*)), simply supported at its ends with respect to lateral bending of its flanges, which is subjected to an external torque T applied at the midspan. Each cross section rotates through an angle ϕ varying along the length of the beam. Also, by virtue of symmetry, there is absolutely no warping of the cross section at the midspan, whereas at each end there is no restraint against warping. Assuming that the web has been removed (Fig. 1(*b*)), the torque T is imposed as a pair of equal and opposite forces $P = T/h$ upon the respective flanges; and, if the beam is not excessively long, the flanges alone are quite capable of supporting the applied torque by virtue of their bending stiffness in their own planes.

Note.—Written comments are invited for publication; the last discussion should be submitted by February 1, 1953.

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Returning to the whole beam of Fig. 1(*a*), the elementary theory of torsion of prismatic bars reveals that at each section some quantity, C, times $d\phi/dx$, of the applied torque is resisted as "simple torsion." The factor C is a property of the cross section and is therefore constant along the beam. Obviously,

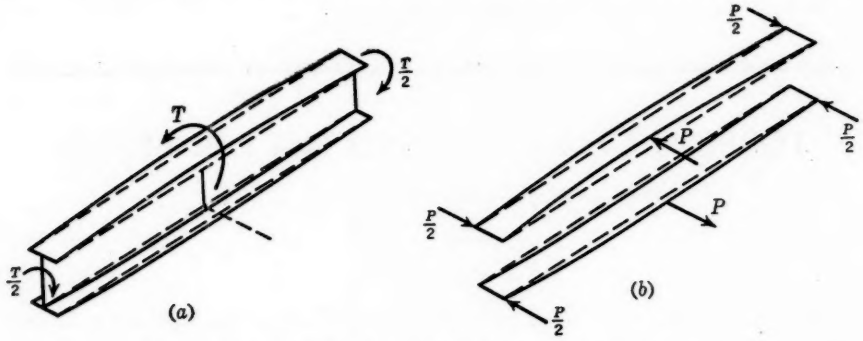

FIG. 1.—RESISTANCE OF COMPLETE BEAM AND OF FLANGES ALONE TO TORSIONAL LOAD APPLIED AT MIDSPAN

$d\phi/dx$ is not constant along the beam but varies from a maximum at the ends to zero at the midspan where the warping is also zero. The net torque at every section, however, has the constant magnitude $T/2$. Therefore, some additional resisting torque must be provided, presumably by the shearing couples in the flanges, so that, at each section, the sum of this resisting torque plus the "simple" torsional resistance, $C\,d\phi/dx$, has the constant total magnitude $T/2$. It may be shown that similar considerations govern the behavior of I-type beams when a torque is applied at any intermediate point. Moreover, any end conditions, symmetrical or unsymmetrical, can be imposed on the beam without annulling the restraint against warping which necessarily exists at the point of application of the torque.

DISPLACEMENT-LOAD RELATIONSHIPS

(*a*) *Simple Torsion.*—In so far as "simple torsion" is concerned, the web and each of the two flanges of, say, a wide flange beam behave very much like three independent narrow rectangular bars. For such a bar, of width b and thickness t, the resisting torque is approximately[2]

$$T = \frac{G}{3} b\, t^3 \left(1 - 0.630\,\frac{t}{b}\right) \frac{d\phi}{dx} \qquad (1a)$$

Considering the ensemble of the web and the two flanges, this portion of the resisting torque may be expressed as

$$T_s = C\,\frac{d\phi}{dx} \qquad (1b)$$

The following formulas for C are suggested because of their simplicity and as

[2] "Theory of Elasticity," by S. Timoshenko, McGraw-Hill Book Co., Inc., New York, N. Y., 1st Ed., 1934, p. 249.

they are sufficiently accurate for practical purposes: For beams with parallel-sided flanges—

$$C = \frac{G}{3}[dt^3_w + 2\,b\,t^3_f] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2a)$$

For beams with tapered flanges—

$$C = \frac{G}{3}[dt^3_w + 2\,b\,(t_1 + t_2)\,(t^2_1 + t^2_2)] \ldots\ldots\ldots\ldots\ldots\ldots (2b)$$

Since d is the total depth of the beam, Eqs. 2 give consideration to the added stiffness developed at the junctions of the web and the flanges by considering the mutual areas twice, and thus tend to compensate for the omission of the correction term $0.630\ t/b$. More accurate formulas for C have been presented by A. A. Griffith and G. I. Taylor,[3,4] by G. W. Trayer and H. W. March,[5] by Inge Lyse, A.M. ASCE, and Bruce G. Johnston,[6] M. ASCE, and by others.

(*b*) *Torsion Bending.*—As noted previously, the twisting of the beam results in lateral displacement of the flanges. Assuming that the displacements are small and that there is no appreciable distortion in the plane of the cross sections, the lateral displacement w of each flange is (see Fig. 2)

$$w = \phi\left(\frac{h}{2}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

in which ϕ is the angular displacement of the cross section and $h/2$ is the distance of the flange centroid from the axis of twist—that is, from the center

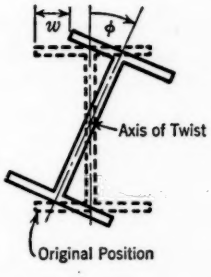

FIG. 2.—ROTATIONAL DISPLACEMENT OF CROSS SECTION

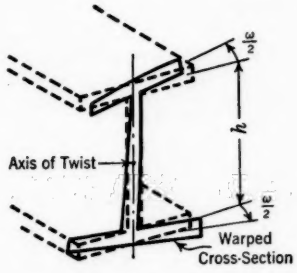

FIG. 3.—PRINCIPAL WARPING OF CROSS SECTION

of the cross section. The elementary theory of bending shows that the bending moment and transverse shear in each flange are

$$M = E\,I_f\,\frac{d^2w}{dx^2} \cong E\,I_f\,\frac{h}{2}\,\frac{d^3\phi}{dx^3} \ldots\ldots\ldots\ldots\ldots\ldots (4a)$$

[3] "The Use of Soap Film in Solving Torsion Problems," by A. A. Griffith and G. I. Taylor, *Reports and Memoranda No. 333*, British Air Ministry, June, 1917.

[4] "The Determination of the Torsional Stiffness and Strength of Cylindrical Bars of Any Sections," by A. A. Griffith and G. I. Taylor, *Reports Memoranda No. 334*, British Air Ministry, June, 1917.

[5] "The Torsion of Members Having Sections Common in Aircraft Construction," by G. W. Trayer and H. W. March, *Report No. 334*, National Advisory Committee for Aeronautics, Washington, D. C., 1929.

[6] "Structural Beams in Torsion," by Inge Lyse and Bruce G. Johnston, *Transactions*, ASCE, Vol. 101. 1936, pp. 857–896.

and

$$S = E I_f \frac{d^3w}{dx^3} \cong E I_f \frac{h}{2} \frac{d^3\phi}{dx^3} \quad \ldots\ldots\ldots\ldots\ldots\ldots (4b)$$

in which I_f is the moment of inertia of one flange about the centroidal axis normal to the flange. The corresponding torque is

$$T_b \cong - S h = - E I_f \frac{h^2}{2} \frac{d^3\phi}{dx^3} \quad \ldots\ldots\ldots\ldots\ldots\ldots (5a)$$

Noting that the moment of inertia of one flange is approximately equal to one half of the moment of inertia of the entire section,

$$T_b \cong - E I_y \frac{h^2}{4} \frac{d^3\phi}{dx^3} \quad \ldots\ldots\ldots\ldots\ldots\ldots (5b)$$

A more precise value for T_b may be obtained by dividing the section into horizontal laminae and subsequently into vertical laminae and evaluating the shear and corresponding torsional moment supported by each lamina. Upon integration in the case of parallel-flanged members, this leads to

$$T_b = - E \left(\frac{h^2 b^3 t_f}{24} + \frac{b^3 t^3 f}{36} + \frac{h^3 t^3_w}{72} \right) \frac{d^3\phi}{dx^3} \quad \ldots\ldots\ldots\ldots (5c)$$

In Eq. 5*c* the first term within the parentheses corresponds to the value given in Eq. 5*a* and the remaining two terms are practically negligible for members of ordinary proportions.

The Differential Equation for Twist

At any section, the total torsion T is equal to

$$T_s + T_b = T \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

Substituting[7] Eqs. 1 and 5*b* into Eq. 6,

$$C \frac{d\phi}{dx} - E I_y \frac{h^2}{4} \frac{d^3\phi}{dx^3} = T \quad \ldots\ldots\ldots\ldots\ldots\ldots (7)$$

If T, the applied torque at the section, is constant in an interval, the general solution of Eq. 7 for that interval is

$$\phi = A_1 \sinh \frac{x}{a} + A_2 \cosh \frac{x}{a} + \frac{T x}{G} + A_3 \quad \ldots\ldots\ldots\ldots\ldots (8)$$

in which

$$a = \sqrt{\frac{E I_y h^2}{4 C}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9)$$

and A_1, A_2, and A_3 are arbitrary constants.

[7] See "Strength of Materials," by S. Timoshenko, D. Van Nostrand Co., Inc., New York, N. Y., Part II, 2d Ed., 1941, p. 285.

Angle of Warp

Consider a given cross section of the beam and imagine a pair of transverse lines (that is, lines parallel to the upper surface) passing through the centroids of each flange (see Fig. 3). Under zero load, these two lines are parallel. Under the action of a torsional moment, however, an angle, ω, will exist

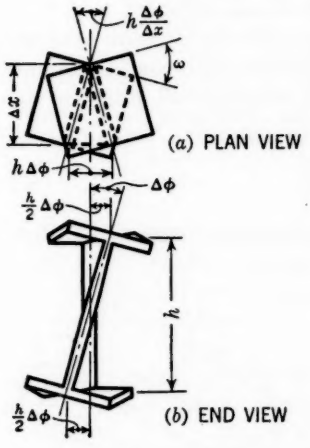

Fig. 4.—Relation Between Warping Angle ω and Twisting Angle ϕ

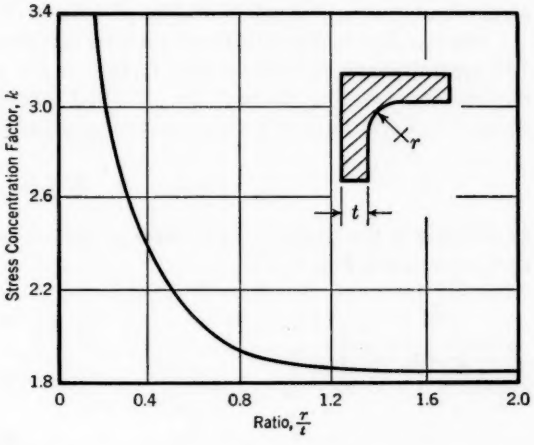

Fig. 5.—Stress Concentration Factors for a Fillet in Single Torsion

between these lines. This angle, ω, the "angle of warp," is indeed a measure of the major nonplanarity or warping of the cross section. Fig. 4 shows that ω is substantially equal to the angle between corresponding edges of the two flanges. Although not rigorous, this definition is adequate for most structural problems.

A value for ω is obtained from the geometrical relationships of Fig. 4:

$$\omega = h\frac{d\phi}{dx} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (10)$$

State of Stress in Torsion

The stresses induced by twisting of I-type sections consist primarily of

1. Shearing stresses due to simple torsion; and
2. Bending and shearing stresses, primarily in the flanges, which are related to the warping resistance.

Shearing Stresses Due to Simple Torsion.—Except for a stress concentration that occurs in the region of the web-to-flange junction, the maximum values of the simple torsional stresses at any cross section occur at the periphery of the section and have the value—

$$\tau_0 = G\,t\frac{d\phi}{dx} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11)$$

—in which t is the local thickness. In the fillet region, this maximum stress is increased to

$$\tau_{max} = k\,\tau_0 \quad \ldots\ldots (12)$$

—in which k is an empirical stress-concentration factor. A generally conservative value of k may be obtained[8] from Fig. 5. It will be remembered that this maximum occurs at the edge of the fillet and that the maximum shearing stress trajectories coincide with the periphery of the cross section.

Stresses Due to Torsion Bending.—In accordance with the relation between the lateral displacement of the flange under torsional load conditions and elementary bending theory, longitudinal bending stresses will exist in the flanges and will have the approximate magnitude—

$$\sigma_b \cong -\frac{E\,h}{2} \times z \times \frac{d^2\phi}{dx^2} \quad \ldots\ldots (13a)$$

in which z is the distance of a point in the cross section from the median line of the web (see Fig. 6).

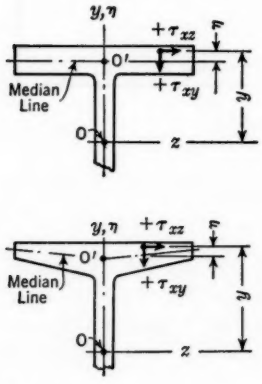

FIG. 6.—COORDINATES AND POSITIVE DIRECTIONS FOR TORSION BENDING STRESSES

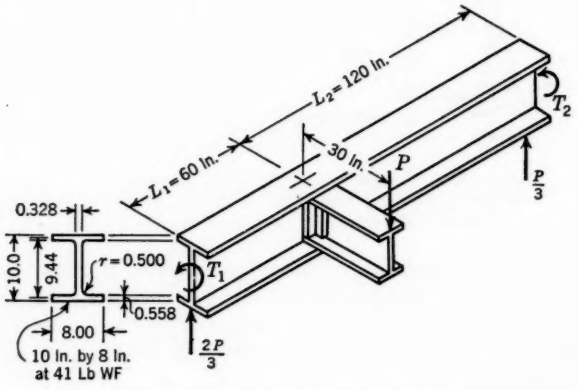

FIG. 7.—BEAM AND LOADING FOR EXAMPLE 1

As indicated previously, the lateral deflections of the flanges due to twist will induce transverse shearing forces,

$$S = E\,I_f\,\frac{h}{2}\,\frac{d^3\phi}{dx^3} \quad \ldots\ldots (13b)$$

in each flange. The corresponding shearing stresses may be assumed to be distributed across the width of the flange in accordance with elementary theory. Thus, for parallel flanges, this distribution will be parabolic and will vary from zero at the edges of the flange to a maximum at the center line of the section ($z = 0$):

$$(\tau_{xz})_b = -\frac{E\,b^3\,t_f\,h}{16}\left(1 - \frac{4\,z^2}{b^2}\right)\frac{d^3\phi}{dx^3} \quad \ldots\ldots (14a)$$

[8] See "Structural Beams in Torsion," by Inge Lyse and Bruce G. Johnston, *Transactions*, ASCE, Vol. 101, 1936, p. 865.

with a maximum value—

$$(\tau_{xy})_b = -\frac{E\,b^3\,t_f\,h}{16}\frac{d^3\phi}{dx^3} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (14b)$$

It may be shown that small additional shearing stresses, τ_{yz}, will also exist in the flanges (see Fig. 6):

$$(\tau_{xy})_b = -\frac{E\,t_f^3\,z}{8}\left(1 - \frac{4\,\eta^2}{t_f^2}\right)\frac{d^3\phi}{dx^3} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (15)$$

Small shearing stresses will also exist in the web, but these, along with τ_{xy} in the flanges, generally may be neglected.

Fundamental Cases of Twisting

The general solution of the differential equation (Eq. 8) contains three arbitrary constants that must be evaluated in terms of the prescribed boundary conditions for a given segment of the beam. Certain fundamental cases of beams or segments subjected to a single torsional moment which is applied at one end and resisted at the other are particularly interesting since a number of practical problems may be solved by the subsequent application of the results of these fundamental studies.

Case I. Flanges Pinned at Each End, Constant Torsion.—In this well-known case, since there can be no restraint against warping at the ends of the beam, the bending moments in the flanges must be zero at these points. The boundary conditions are, therefore, at $x = 0$—$\phi = 0$ and $\frac{d^2\phi}{dx^2} = 0$; and at $x = L$—$\frac{d^2\phi}{dx^2} = 0$. Applying these conditions to the general solution serves to verify the formula for simple twisting of a prismatic bar; thus,

$$\phi = \frac{T}{C}x \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (16a)$$

Since the second and third derivatives of ϕ are zero, it follows from Eqs. 13 and 14 that there are no torsion bending stresses at any section. The total twist is

$$\phi_{(x=L)} = \frac{T\,L}{C} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (16b)$$

Case II. Arbitrary Warping Angle at One End, Flanges Not Restrained at Other End, Constant Torsion.—Let ω_0 be the prescribed warping angle at $x = 0$. Then the boundary conditions for this important case are at $x = 0$—$\phi = 0$ and $\frac{d\phi}{dx} = \frac{\omega_0}{h}$; and at $x = L$—$\frac{d^2\phi}{dx^2} = 0$. These conditions are satisfied when

$$\phi = a\left(\frac{T}{C} - \frac{\omega_0}{h}\right)\left(\tanh\frac{L}{a}\cosh\frac{x}{a} - \sinh\frac{x}{a} - \tanh\frac{L}{a}\right) + \frac{T}{C}x \ldots (17a)$$

The total twist is

$$\phi_{(x=L)} = \frac{T}{C}\left(L - a\tanh\frac{L}{a}\right) + a\frac{\omega_0}{h}\tanh\frac{L}{a} \quad \ldots\ldots(17b)$$

For subsequent use in the calculation of stresses, the necessary derivatives are

$$\frac{d\phi}{dx} = \left(\frac{T}{C} - \frac{\omega_0}{h}\right)\left(\tanh\frac{L}{a}\sinh\frac{x}{a} - \cosh\frac{x}{a}\right) + \frac{T}{C} \quad \ldots\ldots(18a)$$

$$\frac{d^2\phi}{dx^2} = \frac{1}{a}\left(\frac{T}{C} - \frac{\omega_0}{h}\right)\left(\tanh\frac{L}{a}\cosh\frac{x}{a} - \sinh\frac{x}{a}\right) \quad \ldots\ldots(18b)$$

and

$$\frac{d^3\phi}{dx^3} = \frac{1}{a^2}\left(\frac{T}{C} - \frac{\omega_0}{h}\right)\left(\tanh\frac{L}{a}\sinh\frac{x}{a} - \cosh\frac{x}{a}\right) \quad \ldots\ldots(18c)$$

At $x = 0$, the derivatives have the values,

$$\left(\frac{d\phi}{dx}\right)_{(x=0)} = \frac{\omega_0}{h} \quad \ldots\ldots(18d)$$

$$\left(\frac{d^2\phi}{dx^2}\right)_{(x=0)} = \frac{1}{a}\left(\frac{T}{C} - \frac{\omega_0}{h}\right)\tanh\frac{L}{a} \quad \ldots\ldots(18e)$$

$$\left(\frac{d^3\phi}{dx^3}\right)_{(x=0)} = \frac{1}{a^2}\left(\frac{\omega_0}{h} - \frac{T}{C}\right) \quad \ldots\ldots(18f)$$

The case of a beam which is built-in at $x = 0$ and has unrestrained flanges at the opposite end may be obtained directly from Eqs. 17 and 18 merely by setting $\omega_0 = 0$.

Case III. Prescribed Warping Angles at Both Ends, Constant Torsion.— The boundary conditions for this case are at $x = 0$—$\phi = 0$ and $\frac{d\phi}{dx} = \frac{\omega_0}{h}$; and at $x = L$—$\frac{d\phi}{dx} = \frac{\omega_L}{h}$. The particular solution becomes

$$\phi = a\left(\frac{\omega_0}{h} - \frac{T}{C}\right)\sinh\frac{x}{a} + \frac{a\cosh\frac{a}{x}}{\sinh\frac{L}{a}}\left[\frac{\omega_L}{h} - \frac{T}{C} - \left(\frac{\omega_0}{h} - \frac{T}{C}\right)\cosh\frac{L}{a}\right]$$

$$+ \frac{T\,x}{C} - \frac{a}{\sinh\frac{L}{a}}\left(\frac{\omega_L}{h} - \frac{T}{C}\right) + \frac{a\cosh\frac{L}{a}}{\sinh\frac{L}{a}}\left(\frac{\omega_0}{h} - \frac{T}{C}\right) \quad \ldots(19)$$

The total twist is obtained by setting $x = L$ in Eq. 19:

$$\phi_{(x=L)} = \frac{2\,T}{C}\left(\frac{L}{2} - a\tanh\frac{L}{2\,a}\right) + (\omega_0 + \omega_L)\frac{a}{h}\tanh\frac{L}{2\,a} \quad \ldots\ldots(20)$$

For use in the calculation of stresses, the derivatives are

$$\frac{d\phi}{dx} = \left(\frac{\omega_0}{h} - \frac{T}{C}\right)\cosh\frac{x}{a} + \frac{\sinh\frac{x}{a}}{\sinh\frac{L}{a}}\left[\frac{\omega_L}{h} - \frac{T}{C} - \left(\frac{\omega_0}{h} - \frac{T}{C}\right)\cosh\frac{L}{a}\right] + \frac{T}{C} \quad (21a)$$

$$\frac{d^2\phi}{dx^2} = \frac{1}{a}\left(\frac{\omega_0}{h} - \frac{T}{C}\right)\sinh\frac{x}{a} + \frac{\cosh\frac{x}{a}}{a\sinh\frac{L}{a}}\left[\frac{\omega_L}{h} - \frac{T}{C} - \left(\frac{\omega_0}{h} - \frac{T}{C}\right)\cosh\frac{L}{a}\right] \quad (21b)$$

and

$$\frac{d^3\phi}{dx^3} = \frac{1}{a^2}\left(\frac{\omega_0}{h} - \frac{T}{C}\right)\cosh\frac{x}{a} + \frac{\sinh\frac{x}{a}}{a^2\sinh\frac{L}{a}}\left[\frac{\omega_L}{h} - \frac{T}{C} - \left(\frac{\omega_0}{h} - \frac{T}{C}\right)\cosh\frac{L}{a}\right] \quad (21c)$$

At $x = 0$, the derivatives have the values:

$$\left(\frac{d\phi}{dx}\right)_{(x=0)} = \frac{\omega_0}{h} \quad (21d)$$

$$\left(\frac{d^2\phi}{dx^2}\right)_{(x=0)} = \frac{1}{a\sinh\frac{L}{a}}\left[\frac{\omega_L}{h} - \frac{T}{C} - \left(\frac{\omega_0}{h} - \frac{T}{C}\right)\cosh\frac{L}{a}\right] \quad (21e)$$

and

$$\left(\frac{d^3\phi}{dx^3}\right)_{(x=0)} = \frac{1}{a^2}\left(\frac{\omega_0}{h} - \frac{T}{C}\right) \quad (21f)$$

By setting the corresponding values of ω equal to zero, Eqs. 19 to 21 may be reduced to those for beams with one or both ends built-in.

The Fundamental Problem

The stresses and deflections that result from the application of a single torsional load at some point between its two ends constitute the basic problem in the twisting of an I-type beam. Three methods of solution are discussed.

Method I. Indeterminate Analysis.—A uniform beam of length L (see Fig. 7) is subjected to a torsional load T applied at distances L_1 and L_2, respectively, from the ends. The solution is obtained by imagining the beam to be cut at the section where the load is applied. The conditions of equilibrium and continuity at the cut faces are then formulated in four steps: (1) The

warping angles must be consistent; (2) the angular displacements must be equal; (3) the longitudinal fiber stresses must be equal; and (4) the sum of the two resisting torsions must equal the applied torsion.

The method of solution is essentially the same whether the ends of the beam are pinned, built-in, or arbitrarily warped. In fact, the method may be extended to the case where the rotational and warping restraints at the ends of the beam are elastic. The case of a beam with warping angles ω_1 and ω_2 at the ends is considered first and, to make the solution more general, a known difference ϕ_e is assumed to exist between the angular positions at the two ends of the beam—that is, $\phi_2 = \phi_1 + \phi_e$.

It is convenient to take the origin at the cut section for each of the two sublengths, L_1 and L_2, and to take the positive direction for each part as going from the cut section to the respective ends of the beam. Let ω_0 be the warping angle at the cut section associated with the sublengths L_1. Then condition (1) is satisfied, in view of the indicated sign convention, if the warping angle at the cut section associated with L_2 is taken as $-\omega_0$.

Eq. 20 is applied to each sublength to formulate condition (2). Since I_f has the same value at each face of the cut section, condition (3) is formulated by direct application of Eq. 21*e* to both sublengths. The resulting equations are

$$\frac{2\,T_1}{C}\left(\frac{L_1}{2} - a\tanh\frac{L_1}{2\,a}\right) + (\omega_0 + \omega_1)\frac{a}{h}\tanh\frac{L_1}{2\,a} + \phi_e = \frac{2\,T_2}{C}\left(\frac{L_2}{2} - a\tanh\frac{L_1}{2\,a}\right) + (-\omega_0 + \omega_2)\frac{a}{h}\tanh\frac{L_2}{2\,a} \quad \ldots (22a)$$

$$\frac{1}{\sinh\dfrac{L_1}{a}}\left[\frac{\omega_1}{h} - \frac{T_1}{C} - \left(\frac{\omega_0}{h} - \frac{T_1}{C}\right)\cosh\frac{L_1}{A}\right] = \frac{1}{\sinh\dfrac{L_2}{a}}\left[\frac{\omega_2}{h} - \frac{T_2}{C} - \left(\frac{\omega_0}{h} - \frac{T_2}{C}\right)\cosh\frac{L_1}{a}\right] \quad \ldots (22b)$$

and

$$T_1 + T_2 = T \quad \ldots (22c)$$

The simultaneous solution of Eqs. 22 yields values of T_1, T_2, and ω_0 which, by means of Eqs. 18 or Eqs. 21, will determine the shearing and bending stresses throughout the beam.

For the case of a beam with pin-ended flanges, Eqs. 17*b* and 18*e* are used instead of Eqs. 20 and 21*e* to formulate conditions (2) and (3). If $\phi_e = 0$ in the pin-ended case, it will be found that T_1 and T_2 are in inverse proportion to the respective sublengths L_1 and L_2. Thus, ω_0 is the only unknown and this may be determined by either condition (2) or condition (3) using only one equation. The case where L_1 and L_2 are dissimilar beams, joined integrally, can be handled merely by modifying Eq. 22*b*. If the two segments have the same depth, one needs only to multiply each member of Eq. 22*b* by the corresponding value of $E\,I_f$.

The case of applied flange warping moments is also handled by modification of Eq. 22*b*. This case arises, for example, where the flanges of a floor beam are welded directly to the flanges of the girder. If M_f is the value of each of the two equal and opposite warping moments applied, respectively, to the two flanges, one merely writes

$$E I_f \frac{h}{2}\left[\left(\frac{d^2\phi}{dx^2}\right)_1 - \left(\frac{d^2\phi}{dx^2}\right)_2\right] = M_f \quad \ldots\ldots\ldots\ldots\ldots\ldots (23)$$

in which the subscripts indicate evaluation immediately to the left and to the right of the point of application of the external moments, and in which Eq. 18*e* or Eq. 21*e* is used in the formulation.

Method II. Superposition.—Superposition methods have the advantage of avoiding simultaneous equations, but frequently are more time-consuming than direct methods. A typical superposition procedure consists of six steps, as follows:

1. Assume the beam to be cut at the point of application of the torsional load; pick convenient values T'_1 and T'_2 such that $T'_1 + T'_2 = T$.

2. Applying Eq. 18*e* or Eq. 21*e* to each of the two sublengths (note that ω'_0 for one sublength requires $-\omega'_0$ for the other) and equating the results, solve for ω'_0.

3. Using T'_1, T'_2, and ω'_0, calculate the total twist of each segment using Eq. 17*b* or Eq. 19. In general, these twists will not satisfy the condition $\phi_2 = \phi_1 + \phi_e$. Therefore, calculate $\Delta\phi = \phi_2 - \phi_1 - \phi_e$, the error in the total twist.

4. Using either Eq. 16*b* for a pin-ended beam or Eq. 20 for a restrained beam, determine the torsion ΔT that must be applied to the unloaded whole beam of length $L = L_1 + L_2$ to obtain a twist of $-\Delta\phi$.

5. Calculate ω''_0, the correction to ω'_0 due to the added twist $-\Delta\phi$. Note that, in the case of a pin-ended beam, $\omega''_0 = -h\,(\Delta\phi/L)$. In the case of a restrained beam, Eq. 21*a* may be used with $x = L_1$ and $L = L_1 + L_2$.

6. For $T_1 = T'_1 + \Delta T$ and $\omega_0 = \omega'_0 + \omega''_0$, determine stresses and deflections where desired in the first segment; calculate stresses and deflections in the second segment using the corresponding values of torsion and warping.

Method III. Trigonometric Series.—In the case of a pin-ended beam to which a torsional load is applied at $x = c$ (the origin being taken at one end), the angular displacement at any point x may be represented by the series[9]

$$\phi = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L} \quad (n = 1, 2, 3, \cdots) \quad \ldots\ldots\ldots\ldots\ldots (24)$$

in which

$$a_n = \frac{2T \sin\dfrac{n\pi c}{L}}{C\dfrac{n^2\pi^2}{L} + \dfrac{E I_y h^2 n^4 \pi^4}{4L^3}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (25)$$

[9] "On the Application of Trigonometric Series to the Twisting of I-Type Beams," by John E. Goldberg, *Proceedings*, First U. S. National Cong. of Applied Mech., Chicago, Ill., June, 1951 (publication pending).

Because of the powers of n in the denominator, this series converges with reasonable rapidity and only a few terms are required, in general, to obtain satisfactory values for the angular deflections. The series may be differentiated to obtain the functions necessary to the calculation of stresses:

$$\frac{d\phi}{dx} = \sum_{n=1}^{\infty} a_n \frac{n\pi}{L} \cos \frac{n\pi x}{L} \quad \ldots\ldots (26a)$$

and

$$\frac{d^2\phi}{dx^2} = -\sum_{n=1}^{\infty} a_n \frac{n^2\pi^2}{L^2} \sin \frac{n\pi x}{L} \quad \ldots\ldots (26b)$$

The case of several torsional loads may be obtained from Eqs. 26 by superposition. For cases in which restraints against warping exist at the ends of the beam, the final solution may be obtained by superimposing upon the results of Eq. 24 a residual solution from Case III which cancels the untenable part of the warping displacements at the ends of the beam.

In the case of a pair of equal and opposite warping moments M_f applied to the flanges of a pin-ended beam at $x = c$, the corresponding series are

$$\phi = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \; (n = 1, 2, 3, \cdots) \quad \ldots\ldots (27a)$$

$$\frac{d\phi}{dx} = \sum_{n=1}^{\infty} b_n \frac{n\pi}{L} \cos \frac{n\pi x}{L} \quad \ldots\ldots (27b)$$

$$\frac{d^2\phi}{dx^2} = -\sum_{n=1}^{\infty} b_n \frac{n^2\pi^2}{L^2} \sin \frac{n\pi x}{L} \quad \ldots\ldots (27c)$$

in which

$$b_n = \frac{M_f \frac{h}{L} \frac{n\pi}{L} \cos \frac{n\pi c}{L}}{C \frac{n^2\pi^2}{L} + E I_y \frac{h^2}{4} \frac{n^4\pi^4}{L^3}} \quad \ldots\ldots (27d)$$

Unlike the previous methods, the series method does not require decomposition of the beam into two sublengths.

Illustrative Examples

Example 1. Pin-Edged Beam with Torsion Applied at Intermediate Point of Span (Fig. 7).—As indicated previously, it can be shown that an applied torsional load is distributed to the two ends of a uniform pin-ended beam in inverse proportion to the sublengths. Thus, T_1 and T_2 are immediately known:

$$T_1 = \frac{T L_2}{L_1 + L_2} = \frac{T L_2}{L} = \frac{2T}{3} \quad \ldots\ldots (28a)$$

and

$$T_2 = \frac{T L_1}{L} = T - T_1 = \frac{T}{3} \quad \ldots\ldots (28b)$$

Therefore, it is necessary to determine only, say, the warping angle at the point of loading. For the beam in Fig. 7: $I_y = 47.7$ in.^{4}; $E = 29 \times 10^6$ lb per sq in.; $G = 11 \times 10^6$ lb per sq in.; $C = G/3\ (2 \times 8.00 \times 0.558^3 + 10.00 \times 0.328^3) = 11.484 \times 10^6$ lb-in.^{2}; and $a = \frac{9.44}{2}\sqrt{\frac{29 \times 10^6 \times 47.7}{11.484 \times 10^6}} = 51.80$ in.

Finally,

$$L_1 = 60 \text{ in.}; \tanh \frac{L_1}{a} = 0.8202; a \tanh \frac{L_1}{a} = 42.49 \text{ in.}; L_1 - a \tanh \frac{L_1}{a} = 17.51 \text{ in.}$$

and

$$L_2 = 120 \text{ in.}; \tanh \frac{L_2}{a} = 0.9808; a \tanh \frac{L_2}{a} = 50.80 \text{ in.}; L_2 - a \tanh \frac{L_2}{a} = 69.20 \text{ in.}$$

Noting that $+\omega_0$ to the left of the load implies $-\omega_0$ to the right (if x is positive both directions from the load) and applying Eq. 17*b* to each sublength and equating the total twists,

$$\frac{T_1}{C}(17.51) + 42.49\frac{\omega_0}{h} = \frac{T_2}{C}(69.20) - 50.80\frac{\omega_0}{h} \quad \ldots\ldots\ldots (29)$$

Since $T_1 = 2\,T_2 = 2\,T/3$, Eq. 29 yields

$$\frac{\omega_0}{h} = 0.1221\frac{T}{C} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (30)$$

Eq. 18*e* may be used instead of Eq. 17*b*:

$$\frac{1}{a}\left(\frac{T_1}{C} - \frac{\omega_0}{h}\right)0.8202 = \frac{1}{a}\left(\frac{T_2}{C} + \frac{\omega_0}{h}\right)0.9808$$

which yields the same value of ω_0/h.

By Eq. 17*b* the angular displacement at the point of loading is

$$\phi = \frac{2\,T}{3\,C}(17.51) + 42.49\left(0.1221\frac{T}{C}\right) = 16.86\frac{T}{C} \quad \ldots\ldots\ldots (31)$$

For the calculation of stresses in the shorter span adjacent to the load (that is, at $x = 0$), the following derivatives are required: By Eqs. 18—

$$\frac{d\phi}{dx} = 0.1221\frac{T}{C} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (32a)$$

$$\frac{d^2\phi}{dx^2} = \frac{1}{51.80}\left(\frac{2\,T}{3\,C} - 0.1221\frac{T}{C}\right)0.8202 = 0.00866\frac{T}{C} \quad \ldots\ldots\ldots (32b)$$

and

$$\frac{d^3\phi}{dx^3} = \frac{1}{(51.80)^2}\left(0.1221\frac{T}{C} - \frac{2\,T}{3\,C}\right) = -0.000204\frac{T}{C} \quad \ldots\ldots\ldots (32c)$$

and by Eq. 6—

$$T_b = T_1 - T_s = \frac{2\,T}{3} - C\frac{d\phi}{dx} = 0.5446\,T \quad \ldots\ldots\ldots\ldots (33a)$$

Check of Eq. 33*a* by Eq. 5*b*:

$$T_b = -E\,I_y\frac{h^2}{4}\frac{d^3\phi}{dx^3}$$

$$= -29 \times 10^6 \times 47.7 \times \frac{9.44^2}{4}\left(-\frac{0.000204\,T}{11.484 \times 10^6}\right) = 0.547\,T \quad \ldots (33b)$$

Stresses in the flange immediately to the left of the applied load are determined, as follows, by Eq. 11:

$$\tau_0 = G\,t\frac{d\phi}{dx} = 11 \times 10^6 \times 0.558 \times 0.1221\,\frac{T}{11.484 \times 10^6} = 0.0652\,T \quad \ldots (34)$$

The effect of the stress concentration at the fillet is determined as follows: $r/t = 0.93$; $k = 1.92$ (from Fig. 5); and $\tau_{s(\max)} = 1.92 \times 0.0652\,T = 0.125\,T$. The shear per flange is

$$S = \frac{T_b}{h} = \frac{0.5446\,T}{9.44} = 0.0577\,T \quad \ldots\ldots\ldots\ldots\ldots (35a)$$

and

$$\tau_{b(\max)} = \frac{3\,S}{2\,A_f} = \frac{3 \times 0.0577\,T}{2 \times 0.558 \times 8.00} = 0.0194\,T \quad \ldots\ldots\ldots (35b)$$

The longitudinal fiber stresses at the edges of the flanges are by Eq. 13*a*

$$\sigma_b \cong 29 \times 10^6 \times \frac{9.44}{2} \times 4.00 \times \frac{0.00866\,T}{11.484 \times 10^6} = 0.413\,T \quad \ldots\ldots (35c)$$

Thus, if $P = 2{,}000$ lb, then for the beam in Fig. 7, $T = 60{,}000$ lb-in., and (in pounds per square inch): $\tau_{s(\max)} = 7{,}500$; $\tau_b = 1{,}164$; and $\sigma_b = 24{,}780$.

These stresses, of course, are to be added to the stresses due to vertical shear and bending.

In this example, the total deformations are small and, therefore, the change in the applied torsion is practically negligible. Likewise, if P remains vertical, its component in the lateral direction of the beam is also negligible. Such effects, however, should be checked in general and, if necessary, corrections can be introduced on the basis of the deformed geometry.

In addition to the foregoing procedure, Example 1 can be solved by the method of superposition or by the trigonometric series method. Two terms of the series Eq. 26 gives $\phi = \dfrac{16.64\,T}{C}$ at the load point.

Example 2. Beam with Prescribed Warping Angles at Ends, Torsion Applied at Intermediate Point.—A beam similar to that of Example 1, but with prescribed warping angles, ω_1 and ω_2, at the left and right ends, respectively, is analyzed. The distribution of the applied torsion to the two sublengths of the

beam is not a direct function of the lengths as in the case of the pin-ended beam; thus,

$$\tanh \frac{L_1}{2\,a} = \tanh \frac{60}{103.6} = 0.5221; \text{ and } a \tanh \frac{L_1}{2\,a} = 27.05$$

$$\tanh \frac{L_2}{2\,a} = \tanh \frac{120}{103.6} = 0.8202; \text{ and } a \tanh \frac{L_2}{2\,a} = 42.49 \text{ in.}$$

$$\left(\frac{L_1}{2} - a \tanh \frac{L_1}{2\,a}\right) = 2.96 \text{ in.; and } \left(\frac{L_2}{2} - a \tanh \frac{L_2}{2\,a}\right) = 17.51 \text{ in.}$$

$$\sinh \frac{L_1}{a} = 1.4354; \text{ and } \cosh \frac{L_1}{a} = 1.7494$$

$$\sinh \frac{L_2}{a} = 5.0223; \text{ and } \cosh \frac{L_2}{a} = 5.1209.$$

Let ω_0 be the warping angle for segment L_1 immediately to the left of the applied load. Then, taking x as also positive in L_2, the warping angle for this sublength at the load ($x = 0$) is $-\omega_0$.

Eqs. 22, which equate the angular displacements of the two sublengths and the flange bending stresses at the load point, and insure equilibration of the applied load, become

$$\frac{2\,T_1}{C}\,2.96 + \frac{\omega_0 + \omega_1}{h}\,27.05 + \phi_e = \frac{2\,T_2}{C}\,17.51 + \frac{-\omega_0 + \omega_2}{h}\,42.49 \quad \ldots (36a)$$

$$\frac{1}{1.4354}\left(\frac{\omega_1}{h} - 1.7494\,\frac{\omega_0}{h} + 0.7494\,\frac{T_1}{C}\right) = \frac{1}{5.0223}\left(\frac{\omega_2}{h} + 5.1209\,\frac{\omega_0}{h} + 4.1209\,\frac{T_2}{C}\right) \quad \ldots (36b)$$

and

$$\frac{T_1}{C} + \frac{T_2}{C} = \frac{T}{C} \quad \ldots (36c)$$

The simultaneous solution of Eqs. 36 yields

$$\frac{T_1}{C} = 0.732\,\frac{T}{C} - 0.589\,\frac{\omega_1}{h} + 0.589\,\frac{\omega_2}{h} - 0.01210\,\phi_e \quad \ldots (37a)$$

$$\frac{T_2}{C} = 0.268\,\frac{T}{C} + 0.589\,\frac{\omega_1}{h} - 0.589\,\frac{\omega_2}{h} + 0.01210\,\phi_e \quad \ldots (37b)$$

and

$$\frac{\omega_0}{h} = 0.726\,T - 0.042\,\frac{\omega_1}{h} + 0.264\,\frac{\omega_2}{h} - 0.0073\,\phi_e \quad \ldots (37c)$$

For the case of the beam with built-in ends, it is necessary only to set ω_1 and ω_2 equal to zero.

This problem can be solved by a method of superimposing a residual solution for the proper end conditions upon the solution for the pin-ended case which was obtained in Example 1. By substituting the values of T_1 and ω_0 into Eq. 18*b* and evaluating at $x = L_1$, it is found that $\omega/h = 0.3491\ T/C$ at

the left end of the pin-ended beam. Similarly, using T_2 and $-\omega_0$ with $x = L_2$, it is found that $\omega/h = 0.2444\ T/C$ at the right end. Therefore, in order to adapt the pin-ended solution to the prescribed end conditions, the designer must impose additional warping angles ω_l and ω_r at the left and right ends of the beam such that

$$\frac{\omega_l}{h} = \frac{\omega_1}{h} - 0.3491\,\frac{T}{C} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (38a)$$

and

$$\frac{\omega_r}{h} = \frac{-\omega_2}{h} + 0.2444\,\frac{T}{C} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (38b)$$

The additional torque ΔT induced by imposing these angles along with the prescribed differential displacement of the ends, ϕ_e, is determined by Eq. 20:

$$\phi_{(x=L)} = \frac{2\,\Delta T}{C}\left(\frac{L}{2} - a\tanh\frac{L}{2\,a}\right) + (\omega_l + \omega_r)\,\frac{a}{h}\tanh\frac{L}{2\,a} \quad \ldots\ldots (39a)$$

or

$$-\phi_e = \frac{2\,\Delta T}{C}\left(\frac{180}{2} - 51.80\tanh\frac{180}{103.60}\right) + \left(\frac{\omega_1}{h} - 0.3491\,\frac{T}{C} - \frac{\omega_2}{h} + 0.2444\,\frac{T}{C}\right) 51.80\tanh\frac{180}{103.60} \quad \ldots (39b)$$

Eq. 39 yields

$$\frac{\Delta T}{C} = 0.0617\,\frac{T}{C} - 0.589\,\frac{\omega_1}{h} + 0.589\,\frac{\omega_2}{h} - 0.0121\,\phi_e \quad \ldots\ldots\ldots (40a)$$

Adding and subtracting to $\frac{T_1}{C}$ and $\frac{T_2}{C}$ for the pin-ended case results in

$$\frac{T_1}{C} = \frac{2}{3}\frac{T}{C} + \frac{\Delta T}{C} = 0.729\,\frac{T}{C} - 0.589\,\frac{\omega_1}{h} + 0.589\,\frac{\omega_2}{h} - 0.0121\,\phi_e \quad \ldots (40b)$$

and

$$\frac{T_2}{C} = \frac{T}{3\,C} - \frac{\Delta T}{C} = 0.272\,\frac{T}{C} + 0.589\,\frac{\omega_1}{h} - 0.589\,\frac{\omega_2}{h} + 0.0121\,\phi_e \quad \ldots (40c)$$

—which agree substantially with the simultaneous equation solution.

Example 3. Floor Panel Framing Problem (Fig. 8).—Problems involving interaction of spandrel or floor girders and floor beams may be solved by any one of several procedures. For example, appropriate equations of Cases II and III can be applied to the beam and to each segment of the girder, writing the conditions of twisting and warping equilibrium for each girder-beam junction in terms of the angular and warping displacements, and using the slope deflection equation to express the end moment of the floor beam. Indeed, if the floor beam flanges are rigidly attached to the girder flanges, thus forcing the beam to assume the same warping angle as the girder at the junction, it may be necessary to use this procedure. For example, if the beam and girder have the same depth, the equilibrium condition for flange warping moments

consists of multiplying Eq. 18*e* or Eq. 21*e* by the corresponding $(E\,I_f)$-values and setting the sum of the three values thus obtained at each joint equal to zero.

In general, however, the connections between beam and girder will be such that the beam is free to warp at the junction, and the following simple pro-

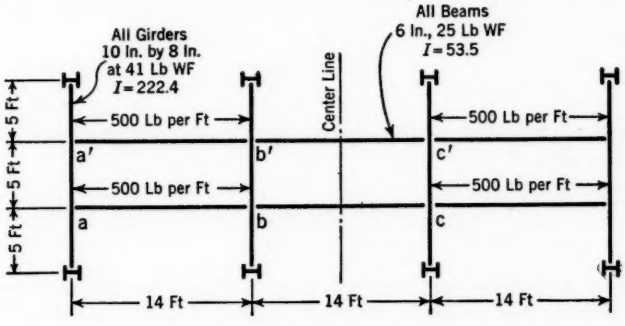

FIG. 8.—FLOOR FRAMING PLAN FOR EXAMPLE 3

cedure will be adequate. This procedure requires knowledge of the angle of twist of the girder at each junction for unit torques applied at these points. The data are obtained by preliminary solutions for the girder for a series of unit torques, following the method of Example 1 or by the series method. Thus, using the results of Example 1, a torque T_a applied at the third-point results in a rotation at that point having a value

$$\phi'_a = \frac{T_a}{C}\,16.86 \qquad (41a)$$

This torque also induces a rotation at point b which is obtained by evaluating Eq. 17*a* at $x = 60$ in. in the 120-in. sublength:

$$\phi'_{a'} = \frac{T_a}{C}\,16.86 - \left[a\left(\frac{T_2}{C} - \frac{\omega_0}{h}\right)\left(\tanh\frac{L_2}{a}\cosh\frac{x}{a} - \sinh\frac{x}{a} - \tanh\frac{L_2}{a}\right) + \frac{T_2}{C}\,x\right] \qquad (41b)$$

Substituting the values $\frac{\omega_0}{h} = -\,0.1221\,\frac{T_a}{C}$, $T_2 = \frac{T_a}{3}$, $\cosh\frac{x}{a} = 1.7494$, $\sinh\frac{x}{a} = 1.4354$, and $\tanh\frac{L_2}{a} = 0.9808$, gives

$$\phi'_{a'} = \frac{T_a}{C}\,13.38 \qquad (41c)$$

It may be noted that using only two terms of the series of Eq. 24 gives $\phi'_{a'} = 13.48\ T/C$. By Maxwell's theorem, the total angular displacement at

$x = a$ is

$$\phi_a = \frac{T_a}{C} 16.86 + \frac{T_{a'}}{C} 13.38 \ldots\ldots\ldots\ldots\ldots\ldots (41d)$$

In the general case, it is necessary to obtain similar expressions for each junction and to adjust for relative rotation at the ends of the girders. In the panel shown, by virtue of symmetry and assuming no rotation at the ends of the girders,

$$\phi_a = \frac{T_a}{C} 30.24 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (41e)$$

The bending moment at the end of the floor beam is given by the slope deflection equation,

$$M_{ab} = M_F + \frac{2 E I}{L_{ab}} \left(\frac{3 y_b - y_a}{L_{ab}} - 2 \phi_a - \phi_b \right) \ldots\ldots\ldots\ldots (42)$$

Although the method may be used in cases where the distortion of the beam-girder junction causes a definite angular difference between the rotation of the girder and the end slope of the beam, in this example it is assumed that $\theta_a = \phi_a$, and similarly at other junctions.

To include the vertical deflections, y_a and y_b, as unknowns would require several additional simultaneous equations. From a practical standpoint, of course, it is desirable to avoid the added complications of the additional equations. Fortunately, because of the torsional flexibility of the girder and the relative insensitivity of the beam reactions to the end moments, it is generally possible to treat the vertical deflections as known quantities that have been determined previously by estimate or by preliminary analysis.

Thus, for the loading of Fig. 8, one may estimate that beam ab is pin-ended at point a and is approximately 50% restrained at point b (primarily by span bc). Comparing with the case of a beam fully restrained at point b and hinged at point a, for which the end shears are 5 $W/8$ and 3 $W/8$, a reasonable estimate of the end shears in this problem would be 9 $W/16$ and 7 $W/16$. The difference, $W/8 = 875$ lb, would cause a differential vertical deflection of the girders (assuming fixed-end conditions), $y_b - y_a = 0.0047$ in. The contribution of this term to the end moments is $\frac{2 E I}{L_{ab}} \left(3 \frac{y_b - y_a}{L_{ab}} \right) = \frac{2 \times 30 \times 10^6 \times 53.5}{168}$ $\times 3 \times \frac{0.0047}{168} = +1{,}605$ lb-in. at each end of the beam.

The moments at joint a are

$$T_a = (C/30.24)\, \phi_a = -(11.484 \times 10^6/30.24)\, \phi_a = -0.3798 \times 10^6\, \phi'_a \ldots (43a)$$

and

$$M_{ab} = \frac{500 \times 14^2 \times 12}{12} + 1{,}605 - \frac{2 \times 30 \times 10^6 \times 53.5}{168} (2 \phi_a + \phi_b) \ldots (43b)$$

and, since the joint is in equilibrium,

$$\sum M_a = T_a + M_{ab} = 99{,}605 - 38.594 \times 10^6\, \phi_a - 19.107 \times 10^6\, \phi_b = 0 \ldots (43c)$$

At joint b, since $\phi_c = -\phi_b$,

$$T_b = -(C/30.24)\,\phi_b = -0.3798 \times 10^6\,\phi_b \quad \ldots\ldots\ldots (44a)$$

$$M_{ba} = -\frac{500 \times 14^2 \times 12}{12} + 1{,}605 - \frac{2 \times 30 \times 10^6 \times 53.5}{168}(2\,\phi_b + \phi_a) \quad \ldots (44b)$$

$$M_{bc} = \frac{2 \times 30 \times 10^6 \times 53.5}{168}\,\phi_b \quad \ldots\ldots\ldots\ldots\ldots (44c)$$

and

$$\sum M_b = T_b + M_{ba} + M_{bc}$$
$$= -96{,}395 - 57.701 \times 10^6\,\phi_b - 19.107 \times 10^6\,\phi_a = 0 \quad \ldots (44d)$$

The solution of the two equilibrium equations (Eqs. 43*c* and 44*d*), using either algebraic methods or successive approximations, is $\phi_a = 4.0761 \times 10^{-3}$ radians; and $\phi_b = -3.0203 \times 10^{-3}$ radians. The corresponding torques and moments are, at joint a (Fig. 8)—$T_a = -1{,}550$ lb-in.; and $M_{ab} = 1{,}550$ lb-in.; and at joint b—$T_b = 1{,}150$ lb-in.; $M_{ba} = -58{,}860$ lb-in.; and $M_{bc} = 57{,}710$ lb-in.

In place of the estimated difference of 875 lb in the end shears of beam ab, the corrected value is $\frac{2(58{,}860 - 1{,}550)}{168} = 684$ lb which would cause a differential vertical deflection of 0.0037 in. in the girders. If the slope deflection analysis were carried through, the error of 0.0010 in., in turn, would cause $\Delta T_a = 3$ lb-in.; $\Delta M_{ab} = -3$ lb-in.; $\Delta M_{ba} = -70$ lb-in.; $\Delta M_{bc} = +68$ lb-in.; and $\Delta T_b = +2$ lb-in. These results are sufficiently small to require no further correction, and, therefore, $T_a = -1{,}547$ lb-in.; $M_{ab} = +1{,}547$ lb-in.; $T_b = 1{,}152$ lb-in.; $M_{ba} = -58{,}930$ lb-in.; and $M_{bc} = +57{,}768$ lb-in.

It will be noted that the analysis could be made by the moment distribution method instead of the slope deflection procedure. It will be noted also that a separate analysis for the effects of a unit differential vertical deflection of the girders would be helpful in some cases, particularly if several successive corrections to the vertical deflections are required.

For the case in which beams bc are also loaded with a distributed load of 500 lb per ft, the designer can take three eighths of the total load on span ab as an initial estimate for the girder load at point a, whereas the girder load at point b is estimated at five eighths of the load on span ab plus one half of the load on span bc. The difference between these reactions is 5,250 lb, which causes a differential vertical deflection of 0.0283 in. Carrying through the procedure as before, $T_a = -1{,}230$ lb-in.; $M_{ab} = 1{,}230$ lb-in.; $T_b = 343$ lb-in.; $M_{ba} = -115{,}625$ lb-in.; and $M_{bc} = 115{,}282$ lb-in.

The corrected difference between reactions at points b and a is 4,864 lb. The reduction—5,250 − 4,864 = 386 lb—results in the following corrections: $\Delta T_a = +5$; $\Delta M_{ab} = -5$; $\Delta T_b = +3$; $\Delta M_{ba} = -145$; and $\Delta M_{bc} = +142$ lb-in. Obviously, further corrections are negligible.

Conclusion

The principles of a linear torsion bending theory for I-type and H-type sections have been reviewed and have been applied to certain idealized forms

of problems which occur in structural practice. Although a small deflection theory, it is felt that this viewpoint is valid for most cases that occur in practice. The extent to which the actual problem may be formulated directly in terms of this theory depends on such physical factors as the true continuity and rigidity of connections and the restraints imposed by slabs, walls, and fireproofing. The evaluation of these factors and the determination of the extent to which the idealized solution is valid are the functions of the engineer.

The concept of the warping angle ω has been introduced as a tangible parameter in the belief that its easily visualized physical character leads to a clearer understanding of the mode of deformation and hence a simpler formulation of the problems lying within the scope of this theory.

Several simple problems have been solved to illustrate the basic principles. By no means do these cover the field of applicability of these principles. Many additional types of problems can be solved by this procedure and, indeed, all such problems must satisfy these basic equations.

In these equations, the end conditions have been prescribed arbitrarily and may therefore be expressed in terms of elastic parameters which will then represent the coupling or interaction forces between the twisted member and adjacent parts of the frame. The possibility of working this concept into the slope deflection or moment distribution procedure is obvious.

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